Proof that stable monotonic equilibrium distributions in a continuous-focusing channel are necessarily axisymmetric

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The transverse Vlasov equilibrium distribution function of an unbunched ion beam propagating in a continuous-focusing channel is specified by a function $f_\perp(H_\perp)$, where $H_\perp$ is the single-particle Hamiltonian. In standard treatments of continuous-focusing equilibria in Vlasov-Poisson electrostatic models, it is assumed that a stable beam equilibrium specified by monotonic $f_\perp(H_\perp)$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ is axisymmetric (no variation in azimuthal angle, i.e., with $\partial/\partial\theta = 0$). In this paper a simple, but rigorous, proof is presented that *only* axisymmetric equilibrium solutions are possible in Vlasov-Poisson models for any physical choice of $f_\perp(H_\perp)$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ if the confining boundary of the system (the beam pipe) is axisymmetric or if the geometry is radially unbounded.

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The continuous-focusing model with a linear applied focusing force has been extensively studied by Davidson [1,2] and Reiser [3], and broad reviews can be found in U.S. Particle Accelerator School courses [4]. Although the model can only be regarded as a highly idealized representation of more realistic periodic focusing lattices, it is nevertheless useful to illustrate basic physics and scaling properties. The proof we present that a stable, continuous-focusing equilibrium distribution formed from a monotonic decreasing function of the Hamiltonian can only produce axisymmetric beams improves the rigorous understanding of equilibrium properties in the continuous-focusing model. The proof parallels an analysis of a similar equilibrium equation used to model a strongly magnetized, pure electron plasma described by $\mathbf{E}\times\mathbf{B}$ flow in a Penning trap confinement geometry [5].

We consider an infinitely long, unbunched ($\partial/\partial z = 0$) beam of ions of charge q and rest mass m. All particles propagate with axial velocity $\beta_b c = \text{const}$. Here, c is the speed of light *in vacuo*. The beam phase space evolves as a function of the axial coordinate s and is described in terms of the transverse spatial coordinates $\mathbf{x}_\perp$ of the particles and the angles $\mathbf{x}'_\perp$ that the particles make with the axis of the system. We adapt a Vlasov description, where the beam is modeled by a continuous, single-particle distribution function $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$. In the Vlasov description, $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) d^2x_\perp d^2x'_\perp$ represents the (smooth) number of particles within phase-space volume $d^2x_\perp d^2x'_\perp$ at phase-space coordinates $(\mathbf{x}_\perp, \mathbf{x}'_\perp)$ and axial coordinate s. Within the paraxial approximation, $f_\perp$ evolves as an incompressible fluid in 4D transverse phase space according to the nonlinear Vlasov equation [1–4]

$$\left\{\frac{\partial}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial}{\partial \mathbf{x}'_\perp}\right\} f_\perp = 0. \tag{1}$$

Here,

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta0}\mathbf{x}^2_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi \tag{2}$$

is the single-particle Hamiltonian, $k^2_{\beta0} = \text{const} > 0$ is the linear applied focusing constant of the channel, $\gamma_b = 1/\sqrt{1-\beta_b^2} = \text{const}$ is the relativistic gamma factor, and $\phi(\mathbf{x}_\perp, s)$ is the self-field potential generated by the beam space charge. The potential ϕ satisfies the transverse Poisson equation

$$\nabla^2_\perp \phi = -\frac{q}{\epsilon_0}\int d^2x'_\perp f_\perp, \tag{3}$$

with ϕ subject to the appropriate boundary conditions on the transverse machine aperture. Here, ϵ_0 is the permittivity of free space.

The Vlasov-Poisson system given by Eqs. (1)–(3) models the transverse beam evolution in the continuum approximation. The system is solved as an initial value problem where $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ is specified at some initial value of s. The transverse particle Hamiltonian $H_\perp$ given by Eq. (2) is a single-particle constant of the motion with $H_\perp = \text{const}$. Therefore, any function $f_\perp = f_\perp(H_\perp)$ satisfying $f_\perp \geq 0$ will form a valid stationary ($\partial/\partial s = 0$) equilibrium solution to the Vlasov-Poisson system. Functional bounds can be applied to show that the monotonicity condition $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ is a *sufficient* condition for stability of the continuous-focusing equilibrium to both small- and large-amplitude perturbations [1,2,6–8].

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It is convenient to define an effective potential [1–4,9]

$$\psi \equiv \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}. \tag{4}$$

Then $H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \psi$, and without loss in generality, the beam density can be calculated as

$$n = \int d^2 x_\perp' f_\perp(H_\perp) = 2\pi \int_\psi^\infty dH_\perp f_\perp(H_\perp), \tag{5}$$

to recast the Poisson Eq. (3) of the equilibrium as

$$\nabla_\perp^2 \psi = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0 \gamma_b^3 \beta_b^2 c^2} \int_\psi^\infty dH_\perp f_\perp(H_\perp). \tag{6}$$

If the system is confined in a cylindrical, conducting pipe of radius $r = \sqrt{x^2 + y^2} = r_p$ held at potential $\phi = V =$ const, the boundary condition on ψ is

$$\psi(r = r_p) = \frac{1}{2} k_{\beta 0}^2 r_p^2 + \frac{qV}{m\gamma_b^3 \beta_b^2 c^2}. \tag{7}$$

For the special case of an equilibrium with a finite radial extent and line charge $\lambda = q \int d^2 x_\perp n$ in a radially unbounded system (i.e., free space), the boundary condition (7) is replaced by the requirement that

$$\left.\frac{\partial \psi}{\partial r}\right|_{r \gg r_b} = k_{\beta 0}^2 r - \frac{q\lambda}{2\pi\epsilon_0 \gamma_b^3 \beta_b^2 c^2} \frac{1}{r},$$

where $r = r_b$ is the characteristic transverse radius of the equilibrium beam.

Equation (6) is highly nonlinear and the solution ψ must, in general, be numerically constructed for a specific choice of equilibrium function $f_\perp(H_\perp)$. Solutions are analyzed in detail in Refs. [3,4,9] for several choices of $f_\perp(H_\perp)$. In spite of these general difficulties, it is possible to show that Eqs. (6) and (7) admit only axisymmetric [$\partial/\partial\theta = 0$ where $\theta = \tan^{-1}(x, y)$ is the azimuthal angle in plane polar coordinates] solutions for any physical choice of equilibrium function $f_\perp(H_\perp)$. Paralleling Smith *et al.* [5], we assume that a nonaxisymmetric ($\partial/\partial\theta \neq 0$) solution $\psi = \psi_1$ exists to Eqs. (6) and (7). Because the boundary condition (7) is invariant under the rotation, another solution $\psi = \psi_2$ can be generated by actively rotating ψ_1 through any azimuthal angle θ where the solution does not map back onto itself by symmetry. To show these solutions are not supported by contradiction, we first define a positive definite functional

$$F \equiv \int_{\text{pipe}} d^2 x_\perp \left| \frac{\partial}{\partial \mathbf{x}_\perp} (\psi_1 - \psi_2) \right|^2 > 0. \tag{8}$$

Next, we integrate F by parts and apply the divergence theorem with the boundary condition (7) to simplify the result. Then the Poisson equation (6) is applied to equivalently express F as

$$\begin{aligned} F &= -\int_{\text{pipe}} d^2 x_\perp (\psi_1 - \psi_2) \nabla_\perp^2 (\psi_1 - \psi_2) \\ &= \frac{2\pi q^2}{m\epsilon_0 \gamma_b^3 \beta_b^2 c^2} \int_{\text{pipe}} d^2 x_\perp (\psi_1 - \psi_2)[G(\psi_1) - G(\psi_2)], \end{aligned} \tag{9}$$

where

$$G(\psi) \equiv \int_\psi^\infty dH_\perp f_\perp(H_\perp).$$

If $f_\perp$ is a monotonic decreasing function of $H_\perp$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$, then G must also be a monotonic decreasing function of ψ. Thus, if $\psi_1 \geq \psi_2$, then $G(\psi_1) \leq G(\psi_2)$, and conversely, if $\psi_1 \leq \psi_2$, then $G(\psi_1) \geq G(\psi_2)$. Consequently, the integrand in Eq. (9) satisfies $(\psi_1 - \psi_2)[G(\psi_1) - G(\psi_2)] \leq 0$ for all $\mathbf{x}_\perp$ in the pipe, giving $F \leq 0$, which contradicts the requirement that $F > 0$. Therefore, the assumption that a nonaxisymmetric solution exists for a monotonic equilibrium function $f_\perp(H_\perp)$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$ is invalid and any solution to Eqs. (6) and (7) is necessarily axisymmetric.

This proof is easily modified to cover the case of a beam with finite radial extent in free space. Also, the proof is straightforward to generalize to apply to the case where the linear applied radial focusing force $\propto k_{\beta 0}^2 r$ is replaced by a monotonic, nonlinear radial focusing force [i.e., $k_{\beta 0}^2 r^2 \rightarrow f(r)$ with $f(r)$ some positive function satisfying $f(r = 0) = 0$ and $\partial f(r)/\partial r \geq 0$]. The fact that system axisymmetry and stability are connected in simple, continuous-focusing systems is not surprising because the $H_\perp$ depends only on $|\mathbf{x}_\perp'|$ and the spatial $\mathbf{x}_\perp$ and angle $\mathbf{x}_\perp'$ degrees of freedom are strongly connected in an equilibrium. If either the beam pipe is replaced by a nonaxisymmetric conducting pipe, the radial focusing force represented by $k_{\beta 0}^2$ is replaced by nonaxisymmetric ($\partial/\partial\theta \neq 0$) focusing forces, or if $f_\perp(H_\perp)$ is a nonmonotonic function, then no symmetry restrictions can be immediately obtained from the method presented.

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